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SOME DYNAMICS PROBLEMS ON INFLATION OF BUBBLES IN SPACE

by Odus R. Burggraf

Prepared under Contract No. NASw-652 *by*
ASTRO RESEARCH CORPORATION
Santa Barbara, Calif.
for

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LIST OF SYMBOLS

a	film radius
a'	coefficient in Mathieu equation; see Equation (53)
A	constant; also function of time
B	constant; also function of time
c	speed of sound
C	constant; also function of time
D	function of time
g	variable coefficient; see Equation (51)
G	constant; see Equation (7)
H	non-dimensional density; see Equation (22c)
k	$= \frac{\omega}{c}$, wave number
K	constant; see Equation (20)
m	mass of gas in spherical annulus
m_F	mass of film per unit surface area
$\dot{M}$	mass flux of pressurizing gas
M_F	total mass of fluid film
θ	ordering symbol; i.e., $f(z) = \theta(z^n)$ if $\lim\limits_{z \to 0} \frac{f(z)}{z^n} = \text{constant}$
p	pressure
P	non-dimensional pressure; see Equation (22b)
q	dynamic pressure

LIST OF SYMBOLS

q'	coefficient in Mathieu equation; see Equation (53)
Q	volume flux of fluid
r	radius
R	non-dimensional radius; see Equation (22a)
S	spherical harmonic
t	time
T	surface tension
u	radial component of velocity
z	= ka, reduced frequency
	= ωt, independent variable in Mathieu equation, see Equation (53)
α	coefficient in stability equation, see Equation (49)
β	coefficient in stability equation, see Equation (49)
γ	ratio of specific heats of gas
δ	film thickness; also perturbation of surface radius
Δ	transformed amplitude of perturbation of surface radius
ϵ	normalized amplitude of radial oscillation
λ	wave length; also azimuthal angle in spherical coordinates
μ	similarity variable defined by (22d); also micron

LIST OF SYMBOLS

ρ	density
τ	non-dimensional parameter; see Equation (46)
ϕ	velocity potential
ϕ^1	perturbation potential
Φ	velocity potential of unperturbed flow
ψ	function defined by Equation (28b)
ω	frequency in radians/second

Subscripts

0	conditions in gas at inner film radius
1	conditions in fluid at inner film radius
2	conditions in fluid at outer film radius
eq	static equilibrium
F	film
G	gas
i	initial value

Superscripts

$(\dot{\ })$	time derivative
$(\ )'$	derivative with respect to argument of function (eq. 53 excepted)

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I. SUMMARY

The concept of blowing large bubbles from a visco-elastic material has been suggested as a means of constructing enclosures in space. The dynamics of the inflation of such bubbles are investigated here by analyzing a variety of problems. The steady inflation of a spherical bubble is considered first, to determine the radius/time history of a spherical film subject to variable mass flux of pressurizing gas. A self-similar analysis of the gas motion is included. Acoustic oscillations of a bubble about its equilibrium state are considered next with an explicit formula derived for the frequency of the fundamental mode of oscillation. The limiting case of incompressible flow is considered, with the conclusion that it is valid only for thick-walled bubbles. Low frequency instabilities during inflation are considered last. After deriving the fluid dynamical equations, a quasi-steady approximation is carried out for thick-walled bubbles.

Finally, the differential equation is derived for the perturbed motion of a thin-walled bubble as a generalization on the work of Plesset. By means of example the instabilities are shown to be related to the inflation history of the bubble.

II. INTRODUCTION

Conventional structures have little application in large-sized spacecraft design owing to the severe weight penalty imposed on the rocket booster. Thus a variety of structural types have been studied for space application. One such type, the inflatable structure, has the obvious advantages of lightweight and compact storage and has, in fact, been used in actual space operations. In these operations, inflation of the structure produces a large change of volume by inextensible deformations of the structural shell (or membrane), which is stored by folding into a relatively small volume. (The mathematics of folding deformations has been investigated in Ref. 1). Because of the problems associated with the folding operation, alternative approaches are worth exploring. In one such approach, a lump of visco-elastico-plastic material is inflated in a spherically symmetrical manner analagous to blowing a soap bubble.

The bubble material must possess fluid properties to permit the enormous elongation occurring during the inflation process. On the other hand, the final equilibrium state is best maintained as an elastic solid. Materials having these properties are used in many industrial processes involving the "blowing" of thin plastic films. Their non-Newtonian characteristics (viscosity dependent on time, strain, wall thickness, etc.) may be explained by the fact that they actually are solids held in solution. The solvent diffuses to the surfaces and evaporates during inflation. Thus the material becomes more viscous, then plastic, and finally elastic as the solvent evaporates. In addition, strain hardening produces further resistance to flow owing to orientation of the polymer chains in the direction of strain.

Some preliminary experiments with a commercially available solventized Polyvinyl alcohol compound have been conducted. Spheres up to 8" diameter were produced in a moderate vacuum by insertion of atmospheric air into a small droplet of material (Fig. 1). Spheroids up to 6 ft. diameter were made in free air by inflation of larger drops with neutrally buoyant Helium-Nitrogen mixture (Fig. 2). Film thickness of the order of 2μ and of surprising uniformity could be achieved by careful control of the inflation process (although unstable fluctuations were observed under some conditions). The significant result of this experimentation was the confirmation of the initial suspicion that the dynamic stability of the inflation process is a controlling factor in the reduction to practice of this concept.

For successful application, the inflation process must be stable. Since experimental testing in a space environment is excessively costly, analytical studies must be made prior to actual flight tests. The present report is a study of the dynamics of the bubble inflation process. The material properties of the bubble film were assumed to be either purely elastic (as surface tension) or purely fluid, corresponding to late or early times in the inflation process, respectively. During the fluid phase, the motion was assumed to be sufficiently rapid that the inertia terms in the governing equations of motion were dominant, permitting an inviscid analysis. This approach is justified only by the preliminary nature of this study; in a more comprehensive study the effects of the viscous terms should be ascertained.

This report is divided into three parts, in which three basically different problems are analyzed:

1) uniformly expanding bubble,
2) oscillations about the equilibrium state, and
3) stability of the inflation process.

The first part is necessary to provide basic information to determine the time behavior of the fluid properties, to provide input data for the stability analysis, and possibly to provide a basis for a rational design. The second part relates to the bubble behavior at termination of the inflation process, and also provides input data for Part III. The final part then takes up the fundamental question of stability of various phases of the inflation process.

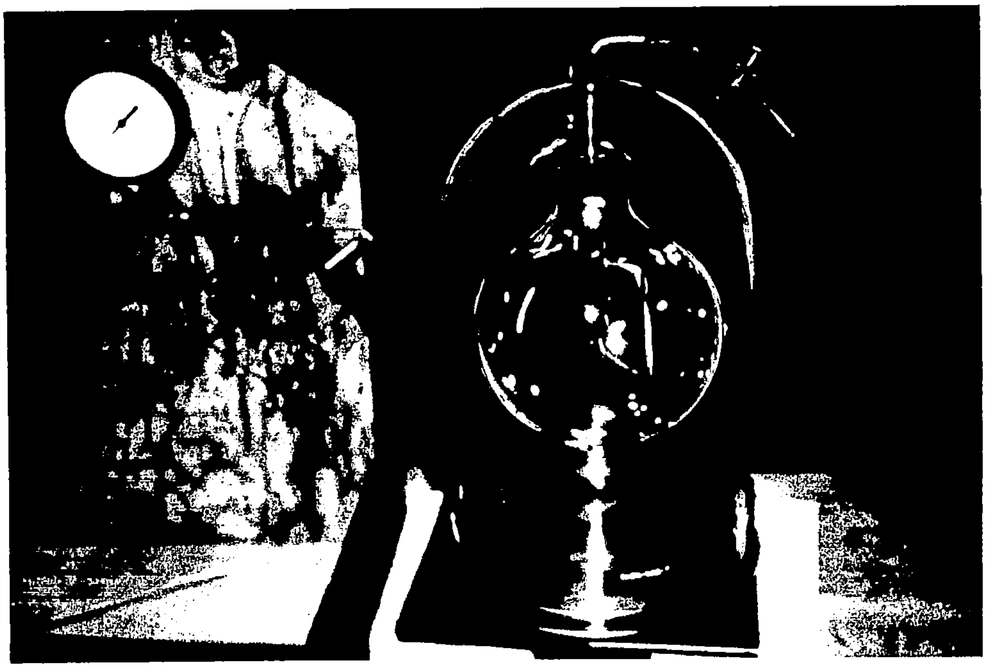

Figure 1. Sphere Blown in Vacuum Bell Jar

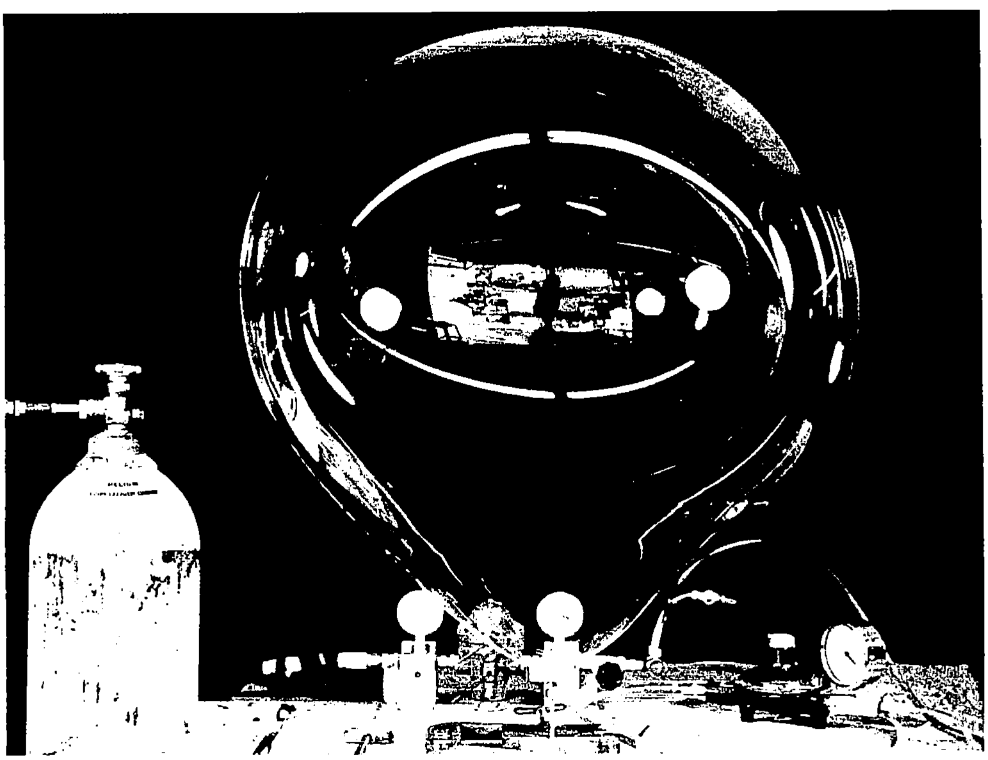

Figure 2. Sphere Blown in Air

III. INFLATION PROCESS

Consider a spherical annulus of material enclosing a volume of gas which may be continuously introduced at the center. The annular film will be assumed to have the properties of an inviscid fluid, including surface tension. Now consider this annulus placed into a vacuum. Owing to the introduction of gas, the bubble expands with time and the film thickness decreases. Hence, initially the mass of the film may be large compared with that of the enclosed gas; after a sufficient time has elapsed, the reverse will be true. In the general case, we see that the dynamics of both gas and fluid film must be considered.

In this section we develop three models for the inflation process. The first model, thick film, light gas, accounts for the fluid flow within the film but neglects any pressure change within the gas; this model should apply to the early stages of the inflation process.

In the second model, thin film, light gas, the flow within the film is neglected also, but the mass of the film is retained as a parameter. This more approximate model cannot be applied at as early a time as the thick-film model, but the additional approximation permits a simpler analysis. Since the density of the film is of the order of 1000 times that of the gas, the thin film model should apply over a wide range of conditions.

The third model, thin film, heavy gas, is developed to ascertain the importance of the momentum of the gas itself, which was neglected in the other models. This model applies at a late time after a large mass of gas has been introduced into the bubble; consequently the mass of the film may be neglected. A self-similar flow analysis is formulated, including surface tension but neglecting the mass of the film with respect to that of the gas.

A. Thick Film, Light Gas Model

Let r_1 be the inner radius and r_2 the outer radius of the fluid film, and u the radial velocity at any radius. Then, regarding the fluid as incompressible, the continuity equation for purely radial, time dependent flow may be written as

$$\frac{d}{dr}(r^2 u) = 0 , \qquad u = \frac{Q(t)}{4\pi r^2} \tag{1a}$$

or

$$Q(t) = \frac{d}{dt}\left(\frac{4}{3}\pi r_1^3\right) = \frac{d}{dt}\left(\frac{4}{3}\pi r_2^3\right) \tag{1b}$$

Here Q(t) has the physical significance of the instantaneous volume flux crossing a sphere of any radius. The pressure in the fluid, assumed inviscid, satisfies the momentum equation

$$\frac{\partial p}{\partial r} = -\rho_F \left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r}\right]$$

$$= -\rho_F \left[\frac{\dot{Q}(t)}{4\pi r^2} - \frac{Q^2(t)}{8\pi^2 r^5}\right]$$

Integrating with respect to r, the pressure distribution is obtained as

$$p(r,t) = p_1(t) + \frac{\rho_F Q(t)}{4\pi}\left(\frac{1}{r} - \frac{1}{r_1}\right) - \frac{\rho_F Q^2(t)}{32\pi^2}\left(\frac{1}{r^4} - \frac{1}{r_1^4}\right) \tag{2}$$

Here $p_1(t)$ is the pressure in the fluid film evaluated at the inner radius r_1. The last term represents the well-known dynamic pressure, and the second term the unsteady component of Bernoulli's equation.

Now let $p_2(t)$ be the pressure in the fluid film at the outer radius and $p_o(t)$ the pressure in the gas at the inner radius of the film. Then if T is the surface tension of the film*, the pressure jump across the film is given by

$$p_2(t) = \frac{2T}{r_2} \tag{3a}$$

$$p_0(t) = p_1(t) + \frac{2T}{r_1} \tag{3b}$$

Evaluating (2) at the outer surface and combining with (3), the gas pressure at r_1 is found:

$$p_0(t) = 2T\left(\frac{1}{r_2} + \frac{1}{r_1}\right) + \frac{\rho_F Q(t)}{4\pi}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - \frac{\rho_F Q^2(t)}{32\pi^2}\left(\frac{1}{r_1^4} - \frac{1}{r_2^4}\right) \tag{4}$$

* The surface tension of the liquid-vacuum interface at r_2 is assumed to be the same as that of the liquid-gas interface at r_1

The inner and outer radii r_1 and r_2 are related by the mass of fluid in the film

$$M_F = \frac{4}{3}\pi\rho_F\,(r_2^3 - r_1^3) \tag{5}$$

or

$$r_2 = \left[\frac{3M_F}{4\pi\rho_F} + r_1^3\right]^{1/3}$$

Hence, with Q(t) given by (1b), a prescribed variation of either r_1 or r_2 allows determination of the gas pressure at the inner film radius r_1. Eliminating Q(t), (4) becomes

$$p_0(t) = 2T\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \rho_F\left\{r_1 r_1 - r_2 r_2 + \frac{3}{2}\left[(r_1)^2 - (r_2)^2\right]\right\} \tag{6}$$

Now assume quasi-steady conditions in the gas subject to isentropic expansion with time; that is, the expansion process is sufficiently slow and the gas sufficiently light that the pressure is essentially uniform throughout the pressurizing gas. Then $p_0 \equiv p_G$ and

$$p_G/\rho_G^{\gamma} = \text{constant}$$

where $\rho_G = M_G/\frac{4}{3}\pi r_1^3$, or

$$\frac{p_0 r_1^{3\gamma}}{M_G^{\gamma}} = G = \text{constant} \tag{7}$$

thus specifying $r_1(t)$ determines $p_o(t)$, which in turn determines $M_G(t)$. These conditions are needed for design of a pressurizing system.

Alternatively, we may specify either $p_o(t)$ or $M_G(t)$ and seek to determine $r_1(t)$. Equation (6) is then a second order, non-linear differential equation for r_1. In this case, numerical or analog procedures are required for the solution. However, it is possible to proceed analytically if the fluid film is thin, as outlined in the following section.

B. <u>Thin Film, Light Gas Model</u>

Denote $\delta = r_2 - r_1$, and consider $\delta \ll r_1 = a$.

From continuity we have

$$r_2 = \left(\frac{r_1}{r_2}\right)^2 r_1$$

$$r_2 = \left(\frac{r_1}{r_2}\right)^2 r_1 + 2\frac{r_1}{r_2}\frac{r_1^2}{r_2}\left[1 - \frac{r_1}{r_2}\right]^3$$

Now in limit $\delta \to 0$, $r_2 \to r_1 \to a$. Hence to first order in δ

$$r_1 r_1 - r_2 r_2 = \frac{\delta}{a}\,[a a - 6 a^2] + \theta(\delta/a)^2$$

$$\frac{3}{2}(r_1^2 - r_2^2) = 6\left(\frac{\delta}{a}\right) a^2 + \theta(\delta/a)^2$$

With these limiting expressions and noting $\rho_F\,\delta \to M_F/4\pi a^2$, equation (6) takes the appropriate form for the thin film approximation:*

* Since this equation is a direct form of F=Ma, it is apparent that equation (14) could be written in a similar form in terms of an effective film radius. However, the force terms would be rather complicated in that case.

$$p_0(t) = \frac{4T}{a} + \frac{M_F}{4\pi}\,\frac{a}{a^2} \tag{8a}$$

$$a + \left(\frac{16\pi T}{M_F}\right) a - \left(\frac{4\pi p_0}{M_F}\right) a^2 = 0 \tag{8b}$$

Here again M_G is given in terms of p_o by (7).

The equilibrium condition is given by $a = 0$:

$$p_0)_{eq} = \frac{4T}{a_{eq}} \tag{9}$$

Now consider $p_o \sim 1/a$; then (8b) becomes a linear equation. Two cases are possible depending on the magnitude of p_o, either greater or less than the equilibrium value. If $p_o > 4T/a$, then a grows exponentially: $a \sim e^{\lambda t}$, $p_0 \sim e^{-\lambda t}$, $M_g \sim e^{(3\gamma - 1/\gamma)\lambda t}$ corresponding to exponentially increasing inflation rate. However, if $p_o < 4T/a$, the bubble radius varies sinusoidally (at least through a half-period): $a \sim \sin \omega t$, $p_0 \sim 1/\sin \omega t$, $M_g \sim (\sin \omega t)^{3\gamma - 1/\gamma}$.

This case cannot correspond to the initial phase of inflation since the growth of the bubble comes about from the inertia of the film (initial conditions), the pressure never exceeding the restoring force of surface tension.

Now let us turn to the non-linear problem. If we regard p_o as a function of a, equation (8b) can be integrated by an inversion of the variables. Writing $\ddot{a} = \dot{a}\frac{d}{da}(\dot{a}) = \frac{d}{da}(\frac{1}{2}\dot{a}^2)$, (8b) becomes

$$\frac{d}{da}\left(\frac{1}{2}\dot{a}^2\right) = \frac{4\pi}{M_F}\, a(p_0 a - 4T)$$

Since the variables are separable, integration is easy:

$$\dot{a} = \left\{ \dot{a}_i^2 - \frac{8\pi}{M_F} \left[2T(a^2 - a_i^2) - \int_{a_i}^{a} p_0 a^2 \, da \right] \right\}^{1/2} \tag{10}$$

Inverting the variables, we obtain

$$t - t_i = \int_{a_i}^{a} \left\{ \dot{a}_i^2 - \frac{8\pi}{M_F} \left[2T(a^2 - a_i^2) - \int_{a_i}^{a} p_0 a^2 \, da \right] \right\}^{-1/2} da \tag{11}$$

Hence if p_o is prescribed (or M_g) the bubble history can be determined by quadrature.

<u>Example 1</u>: Explosive Inflation. Suppose M_g = constant; then the bubble is pressurized initially with the entire mass of gas available.* Then $a_i = 0$, $\dot{a}_i = 0$, and from (7) $p_0 = GM_g^{\gamma} a^{-3\gamma}$.

Hence from (11)

$$t - t_i = \int_{a_i}^{a} \left\{ \frac{16\pi T}{M_F} \left[\frac{GM_g^{\gamma}}{6(\gamma - 1)T} \left(a_i^{-3(\gamma-1)} - a^{-3(\gamma-1)} \right) - (a^2 - a_i^2) \right] \right\}^{-1/2} da$$

Now γ = 5/3 for a monatomic gas, such as helium; for this value of γ the integral can be reduced to standard form:

$$t - t_i = \omega \int_{1}^{(a/a_i)^2} \frac{d\xi}{\sqrt{-\xi^2 + (1+\alpha)\,\xi - \alpha}} \tag{12}$$

$$\alpha = \frac{GM_g^{5/3}}{4\pi T a_i^4} \quad , \quad \omega = 8\sqrt{\frac{\pi T}{M_F}}$$

The integral is recognized as an arc sine; inverting gives the bubble radius versus time for γ = 5/3:

* Note that this condition is incompatible with neglect of pressure variation in the gas when the bubble ultimately becomes very large and very thin. This effect is investigated in the next section.

$$\left(\frac{a}{a_i}\right) = \sqrt{1 + \frac{\alpha - 1}{2}\left[1 - \cos \omega t\right]} \tag{13}$$

From this result the non-linear oscillation of the inviscid bubble is evident. The maximum radius of the bubble is

$$a_{max} = \sqrt{\alpha}\; a_i = \sqrt{\frac{GM_g^{5/3}}{4\pi T a_i^2}} \tag{14}$$

showing that the smaller the initial radius, the larger the maximum radius, as a consequence of the increased work of compression of the gas.

Example 2: Constant Pressure Inflation. The mass flow rate of pressurizing gas may be controlled by a regulator such that the gas pressure remains constant.* In this case equation (11) reduces to

$$(t - t_i) = \sqrt{\frac{3M_F}{8\pi p_0}} \int_{a_i}^{a} \left\{a^3 - \frac{6T}{p_0} a^2 + a_i^3 \left(\frac{6T^2}{p_0 a_i} - 1\right)\right\}^{-1/2} da$$

The cubic in the radicand is easily factored. One root is obviously $a = a_i$; the remaining roots are found by the quadratic formula. Hence

$$t - t_i = \sqrt{\frac{3M_F}{8\pi p_0}} \int_{a_i}^{a} \frac{da}{\sqrt{(a - a_i)(a - b_1)(a - b_2)}}$$

* In a throttling process the parameter G is not constant. For simplicity we assume some pressurizing process which allows G = constant.

where

$$b_{1,2} = \left(\frac{3T}{p_0} - \frac{a_i}{2}\right)\left\{1 \pm \sqrt{1 + \frac{2a_i}{\left(\frac{3T}{p_0} - \frac{a_i}{2}\right)}}\right\} \tag{15}$$

We note that $\left(\frac{3T}{p_0} - \frac{a_i}{2}\right) > 0$ since $a_i < a_{eq} = \frac{4T}{p_0}$.

Defining

$$\sin^2 \phi = \frac{a - a_i}{a - b_1} \quad , \quad k^2 = \frac{b_1 - b_2}{a_i - b_2}$$

the integral may be written in the standard form

$$t - t_i = \sqrt{\frac{3M_F}{2\pi p_0 (a_i - b_2)}} \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$

which is recognized as the incomplete elliptic integral of the first kind.

Inverting the expression, the bubble radius is given in terms of a Jacobi elliptic function (Ref. 2):

$$\frac{a - a_i}{a_i - b_1} = \frac{\sin^2 \lambda(t - t_i)}{1 - \sin^2 \lambda(t - t_i)} = t_n^{\,2} \lambda(t - t_i) \tag{16}$$

where $\lambda = \left[\frac{2\pi p_0 (a_i - b_2)}{3M_F}\right]^{1/2}$. On applying this solution we see immediately that $a \geq a_i$ for $b_1 \leq a_i$ and $a \leq a_i$ for $b_1 \geq a_1$, in order that the functions have real values. The condition $b_1 = a_i$ is equivalent to the pressure condition $p_0 = \frac{4T}{a_i}$, which corresponds to equilibrium at the initial condition. Hence for greater pressure, $b_i < a_i$ and the bubble must expand; for smaller pressure, the reverse is true.

An interesting characteristic of the solution is that the bubble radius becomes infinitely great in finite time; this limiting time is

$$t_{Lim} - t_i = \sqrt{\frac{3M_F}{2\pi p_0(a_i - b_2)}}\ K(k) \tag{17}$$

Since an infinite mass of pressurant must be delivered in this period of time, we see that in practice the constant pressure phase will terminate at a somewhat shorter time, depending on the amount of pressurizing gas available. Following this phase an oscillation of the bubble with constant mass would occur, similar to that treated in Example 1.

C. Thin Film-Heavy Gas Model

In the preceding models of the inflation process, the total mass of the injected gas was assumed to be negligible compared with the fluid film, thus allowing uniform pressure throughout the gas. Since the liquid density is of the order of 1000 times that of the gas, the approximation should be valid over a large range of conditions. However for the applications considered here, the mass of pressurizing gas will ultimately exceed that of the film. In this section, an analysis of the gas motion is formulated and a particular solution is set up.

For the boundary conditions of this problem, it appears natural to formulate the equations using the Lagrangian system (coordinates following the moving particles). We take as independent variables the time t and the mass m, defined as the mass of gas contained between a sphere of arbitrary radius r and the bubble film at $r = a$. Then it is required to determine the dependent variables r, p, ρ as functions of m and t. The equations of motion are continuity, momentum, and a relation between the thermodynamic variables p and ρ.

Continuity $$\frac{\partial r}{\partial m} = -\frac{1}{4\pi\rho r^2} \tag{18}$$

Momentum $$\frac{\partial^2 r}{\partial t^2} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

But $$\frac{\partial p}{\partial m} = \frac{\partial p}{\partial r}\frac{\partial r}{\partial m} = -\frac{1}{4\pi\rho r^2}\frac{\partial p}{\partial r} .$$ Hence

$$\frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} = \frac{\partial p}{\partial m} \tag{19}$$

Isentropic Flow $$p/\rho^{\gamma} = K = \text{constant} \tag{20}$$

In general, these non-linear partial differential equations must be solved by numerical methods. However, under certain conditions, the equations may be reduced to ordinary differential equations, a great advantage in carrying out solutions. In particular, we shall be interested in solutions for which the bubble grows as a power of time.

Appearing in the three equations of motion are five variables r, p, ρ, m, t, having three independent dimensions, mass, length, and time. Since we have three dependent variables, the solutions can be expressed in terms of only one independent variable if all variables can be formed into four independent non-dimensional groups. The theory of dimensional analysis (Ref. 3) ensures that this is possible if only two dimensional constants appear in the problem, in addition to the five variables just mentioned. One of these constants is the isentrope K. For the problem of interest here, we take the surface tension, T, as the remaining constant; hence, we regard the film mass as negligible in this analysis. The four independent non-dimensional groups may be chosen as

$$m/Tt^2, \quad r/At^{+2\gamma/3\gamma-1}, \quad \frac{pt^{2\gamma/3\gamma-1}}{B}, \quad \frac{\rho t^{2/3\gamma-1}}{C}$$

where

$$A = (KT^{\gamma-1})^{-1/3\gamma-1} \tag{21a}$$

$$B = (T^{2\gamma}/K)^{1/3\gamma-1} \tag{21b}$$

$$C = (T^2/K^3)^{1/3\gamma-1} \tag{21c}$$

Hence we seek solutions of the form

$$r = At^{2\gamma/3\gamma-1}\ R(\mu) \tag{22a}$$

$$p = Bt^{-2\gamma/3\gamma-1}\ P(\mu) \tag{22b}$$

$$\rho = Ct^{-2/3\gamma-1}\ H(\mu) \tag{22c}$$

where

$$\mu = \frac{m}{Tt^2} \tag{22d}$$

Solutions of this form are called self-similar, because their form depends only on the similarity variable μ.

Thus

$$(r/a) = R(\mu)/R(0), \quad p/p_0 = P(\mu)/P(0), \quad \rho/\rho_0 = H(\mu)/H(0)$$

When these expressions are substituted into the equations of motion, the explicit time dependence cancels out, leaving a set of ordinary differential equations in terms of the variable μ:

<u>Continuity</u>

$$R^2(\mu)\,\frac{dR}{d\mu} = -\frac{1}{4\pi H(\mu)} \tag{23a}$$

<u>Momentum</u>

$$\frac{1}{2\pi R^2(\mu)}\left\{2\mu^2\,\frac{d^2R}{d\mu^2} - \left(\frac{\gamma+1}{3\gamma-1}\right)\mu\,\frac{dR}{d\mu} - \frac{\gamma(\gamma-1)}{(3\gamma-1)^2}\,R(\mu)\right\} = \frac{dP}{d\mu} \tag{23b}$$

<u>Isentropic Flow</u>

$$P(\mu) = [H(\mu)]^{\gamma} \tag{23c}$$

The boundary condition to be satisfied by the solution must also be expressible in terms of μ alone. Neglecting the film mass, the force balance at the film becomes

$$p_0 = \frac{2T}{a}$$

or in terms of the similarity variables (with M = 0 at the film)

$$P(0) = \frac{2}{R(0)} \tag{23d}$$

The pressure function $P(\mu)$ can be eliminated from equation (23b) to yield a non-linear second order differential equation for $R(\mu)$. Because of the complexity of this equation, numerical methods of solution appear most appropriate. However, certain results are apparent from the form of solution. Thus the bubble radius grows as a power of time

$$a = AR(0)\, t^{2\gamma/3\gamma-1}$$

as does the pressure at the film surface

$$p(0, t) = BP(0)\, t^{-2\gamma/3\gamma-1}$$

The mass flow rate of gas introduced at the center is determined from the condition $R(\mu_{max}) = 0$; hence

$$m_{max} = \mu_{max}\, T t^2$$

Thus, the mass flow rate is linearly increasing with time. Since the acceleration of the gas particles is negative, the pressure must increase from the center outward. This pressure increase is given by

$$p(0, t) - p(m_{max}, t) = B\, [P(0) - P(\mu_{max})]\, t^{-2\gamma/3\gamma-1}$$

Because of the preliminary nature of this study, the differential equations were not integrated to determine the constants in these relations, although it would be desirable to compare these results with those of the simpler theory in the preceding section. A remarkable property of this system of equations is that the solution is completely prescribed by only one boundary condition. Starting the integration at the film surface ($\mu = 0$), the parameter R(o) may be prescribed arbitrarily. However, through equations (23d), (23c), and then (23a) in succession, the derivative R'(o) is uniquely determined. Then by integrating until $R(\mu_{max}) = 0$, the similarity parameter at the center of the sphere is determined.* Thus a single parameter family of solutions is obtained for a given gas.

* This condition may not be satisfied for all initial values.

IV. ACOUSTIC OSCILLATIONS ABOUT EQUILIBRIUM

As a result of the inflation process, the bubble tends to oscillate about its equilibrium state. Spherically symmetric oscillations of large amplitude can be treated by the theory of the preceding section; however, the effects of the gas motion are important at the late stages of development of the bubble and the large amplitude analysis then becomes excessively difficult for the general case. By limiting the analysis to small amplitude oscillations about the equilibrium state, the analysis becomes much simpler and, moreover, is easily generalized to mode shapes which are not spherically symmetric. An acoustic analysis of this type is best suited to slowly inflated bubbles, their low kinetic energy limiting the oscillation to small amplitude.

Acoustic oscillations in a spherical container is a classical problem, dealt with in standard texts. Lamb (Ref. 4) presents an analysis of the characteristic frequencies of vibration of a gas contained in a rigid sphere and in a spherical surface, undergoing radial oscillations. The surface tension boundary condition is dealt with also, for the case of vibrations of an incompressible liquid drop. Hence our analysis will follow those of Lamb, but will include surface tension and mass of the liquid film surrounding a compressible gas. Since we are concerned with the final equilibrium state of the bubble, the film will be assumed to have negligible thickness.

A. General Analysis

The velocity potential ϕ in gas dynamics is defined by

$$\vec{V} = \text{grad}\,\phi \tag{24}$$

In the acoustic approximation ϕ_G satisfies the linear wave equation

$$\nabla^2 \phi_G = \frac{1}{c^2} \frac{\partial^2 \phi_G}{\partial t^2} \tag{25}$$

where C is the speed of sound in the unperturbed gas. If we consider sinusoidal oscillations about equilibrium, we may write

$$\phi_G = \Phi\, e^{i\omega t} \tag{26}$$

where Φ does not depend on time. Then Φ must satisfy the Helmholz equation

$$(\nabla^2 + k^2)\,\Phi = 0 \tag{27}$$

where $k = \omega/c$. Solutions of this equation, in terms of spherical polar coordinates, r, θ, λ, may be expressed in terms of spherical harmonics (see Ref. 4, p. 503); for a solution regular at the origin

$$\Phi = \Sigma\, A_n \psi_n(kr)\, r^n S_n(\theta, \lambda) \tag{28a}$$

where ψ_n is given in terms of Bessel functions of fractional order*

$$z^n \psi_n(z) = \sqrt{\frac{\pi}{2z}}\, J_{n+1/2}(z) \tag{28b}$$

* These functions can be written in closed form in terms of trigonometric functions.

In the linear theory, the boundary conditions may be applied at the unperturbed surface r = a. For the potential given by (28a) the velocity and pressure at the surface are

$$u(a_1 t) = \left[\frac{\partial \phi_G}{\partial r}\right]_{r=a_1} = \Sigma A_n [n\psi_n(ka_1) + ka_1\psi_n'(ka_1)]\, a_1^{n-1}\, S_n(\theta, \lambda)\, e^{i\omega t} \tag{29a}$$

$$\frac{p_O - \bar{p}_G}{\bar{\rho}_G} = -\left[\frac{\partial \phi_G}{\partial t}\right]_{r=a_1} = -i\omega\, \Sigma A_n \psi_n(ka_1)\, a_1^n\, S_n(\theta, \lambda)\, e^{i\omega t} \tag{29b}$$

Note that the symbol $(\bar{\ })$ denotes the unperturbed value. Corresponding to these results, the motion of the inner film surface is given by

$$r_1 - a_1 = \int u(a_1, t)dt = -\frac{i}{\omega}\, \Sigma A_n [n\psi_n(ka_1) + ka_1\psi_n'(ka_1)]\, a_1^{n-1}\, S_n(\theta, \lambda)\, e^{i\omega t}$$

In the fluid film, the velocity potential satisfies Laplace's equation ($c \to \infty$ in incompressible fluid):

$$\nabla^2 \phi_F = 0 \tag{30}$$

The solution for simple harmonic motion in the fluid film is then expressed as the series

$$\phi_F = \Sigma (B_n r^n + C_n r^{-n})\, S_n(\theta, \lambda)\, e^{i\omega t} \tag{31}$$

The velocity and the pressure at any radius are

$$u(r,t) = \frac{\partial \phi_F}{\partial r} = \Sigma (nB_n r^{n-1} - nC_n r^{-n-1})\, S_n(\theta, \lambda)\, e^{i\omega t} \tag{32a}$$

$$\frac{p - \overline{p}_F}{\rho_F} = -\frac{\partial \phi_F}{\partial t} = -i\omega\, \Sigma (B_n r^n + C_n r^{-n})\, S_n(\theta, \lambda)\, e^{i\omega t} \tag{32b}$$

Now the boundary conditions at the inner film surface are continuity of velocity and jump of pressure due to surface tension; following Lamb, the pressure jump condition is

$$p_0 - p_1 = \Delta p_1 = T \left\{ \frac{2}{a_1} + \frac{(n-1)(n+2)}{a_1^2} (r_1 - a_1) \right\}$$

Hence the velocity and pressure matching conditions at $r = r_1$ take the form

$$A_n [n\psi_n(ka_1) + ka\psi_n'(ka_1)] = n[B_n - C_n a_1^{-2n}] \tag{33a}$$

$$\left[B_n + C_n a_1^{-2n} - \frac{\overline{\rho}_G}{\rho_F} A_n \psi_n(ka_1) \right] = -(n-1)(n+2) \frac{T}{\rho_F a_1^3 \omega^2} A_n [n\psi_n(ka_1) + ka_1 \psi_n'(ka_1)] \tag{33b}$$

The third boundary condition is the pressure jump at the outer surface; for an external vacuum

$$p_2 = \frac{2T}{r_2} = T \left\{ \frac{2}{a_2} + \frac{(n-1)(n+2)}{a_2^2} (r_2 - a_2) \right\}$$

with

$$r_2 - a_2 = \int u(a_2, t) = -\frac{i}{\omega} \Sigma\, n(B_n a_2^{n-1} - C_n a_2^{-n-1})\, S_n(\theta, \lambda)\, e^{i\omega t}$$

This condition becomes

$$[B_n + C_n a_2^{-2n}] = n(n-1)(n+2) \frac{T}{\rho_F a_2^3 \omega^2} (B_n - C_n a_2^{-2n}) \tag{33c}$$

Equations (33) are three homogeneous equations for the three unknowns. To have other than the trivial solution, the determinate of the system must vanish, i.e.,

$$\begin{vmatrix} \left(\psi_n + \frac{ka_1}{n}\psi_n'\right) & -1 & \left(\frac{a_1}{a_2}\right)^{-2n} \\ \left\{\left(\psi_n + \frac{ka_1}{n}\psi_n'\right) S_n - \left(\frac{\bar{\rho}_G}{\rho_F}\right)\psi_n\right\} & 1 & \left(\frac{a_1}{a_2}\right)^{-2n} \\ 0 & \left\{1 - \left(\frac{a_1}{a_2}\right)^3 S_n\right\} & \left\{1 + \left(\frac{a_1}{a_2}\right)^3 S_n\right\} \end{vmatrix} = 0$$

where $S_n = n(n-1)(n+2)\dfrac{T}{\rho_F a_1^3 \omega^2}$, and the dependence of ψ_n and ψ_n' on ka_1 is understood. Expanding the determinant gives

$$\left(\psi_n + \frac{ka_1}{n}\psi_n'\right)\left[\left\{1 + \left(\frac{a_1}{a_2}\right)^3 S_n\right\} - \left(\frac{a_1}{a_2}\right)^{-2n}\left\{1 - \left(\frac{a_1}{a_2}\right)^3 S_n\right\}\right]$$
$$+\left\{\left(\psi_n + \frac{ka_1}{n}\psi_n'\right) S_n - \left(\frac{\bar{\rho}_G}{\rho_F}\right)\psi_n\right\}\left[\left\{1 + \left(\frac{a_1}{a_2}\right)^3 S_n\right\} + \left(\frac{a_1}{a_2}\right)^{-2n}\left\{1 - \left(\frac{a_1}{a_2}\right)^3 S_n\right\}\right] = 0 \tag{34}$$

The frequency ω occurs explicitly in S_n and k, and implicitly in ψ_n through its dependence on k.

B. Compressible Bubble of Zero Thickness

For large thin bubbles, the effect of the film mass is negligible compared with that of the gas. In this case we let $a_2 \to a_1 \to a$ and the characteristic equation takes the form

$$\left(\bar{\rho}_G \frac{a^3\omega^2}{T}\right) = 2(n-1)(n+2)\left[n + \frac{ka\psi_n'}{\psi_n}\right] \tag{35a}$$

The speed of sound in the gas is given by

$$c^2 = \frac{\gamma \bar{p}_G}{\bar{\rho}_G}$$

Then with the equilibrium pressure condition* $\bar{p}_G = \frac{4T}{a}$ and the definition of k, we find

$$\bar{\rho}_G \frac{a^3\omega^2}{T} = 4\gamma k^2 a^2 \tag{35b}$$

and the characteristic equation reduces to

$$\frac{(n-1)(n+2)}{2\gamma k^2 a^2}\left[n + \frac{ka\,\psi_n'(ka)}{\psi_n(ka)}\right] = 1 \tag{36}$$

Now let us consider some special cases. The fundamental mode n = 0 is that of purely radial oscillation, and is the one most likely to occur.

Case 1: n = 0. Noting the identity

$$\psi_0(z) = \sqrt{\frac{\pi}{2z}}\, J_{1/2}(z) = \frac{\sin z}{z}$$

* The factor 4 arises because the liquid film has both an inner gas-liquid surface and outer vacuum-liquid surface.

we find

$$\frac{z\,\psi_o'(z)}{\psi_o(z)} = z \cot z - 1$$

Then the characteristic frequencies for the fundamental mode must satisfy the equation

$$\cot z = \frac{1 - \gamma z^2}{z} \tag{37}$$

where z stands for $ka = \frac{\omega a}{c}$. The two sides of this equation are graphed in Figure 3; from the graph it is seen that the roots of the characteristic equation are approximated by $m\pi$, $m = 1, 2, 3,$ the approximation being asymptotically correct for large m.

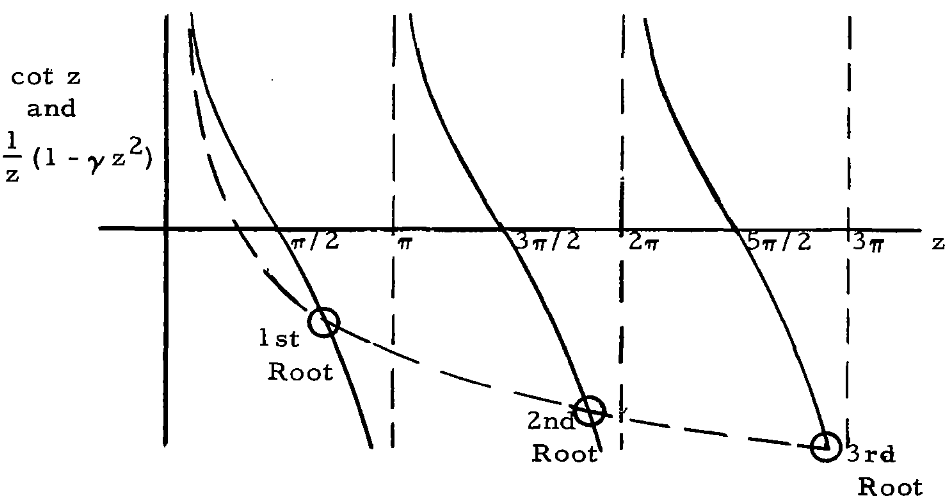

Figure 3 Graph of cot z , solid, and $\frac{1}{z}(1 - \gamma z^2)$, dashed

This fact suggests that an asymptotic expansion is an appropriate procedure for solving the characteristic equation. Also from the graph we see that the iterative procedure

$$z^{(i+1)} = m\pi + \text{arc cot}\,(-\gamma z^{(i)} + \frac{1}{z^{(i)}})$$

beginning with $z^{(1)} = m\pi$, converges to the correct answer. Using this procedure together with the expansion

$$\text{arc cot}\, x = \frac{1}{x} - \frac{1}{3x^3} + \ldots$$

the asymptotic formula for the characteristic frequency is obtained:

$$\frac{\omega a}{c} = m\pi \left[1 - \frac{1}{\gamma m^2 \pi^2} + \ldots\right] \tag{38}$$

This two-term formula is correct to within 1/2 % for m = 1, $\gamma = 1.4$. Obviously it will be much better for m > 1. The wave length of the vibration is

$$\frac{\lambda}{2a} = \frac{1}{m} \left[1 + \frac{1}{\gamma m^2 \pi^2} + \ldots\right] \tag{39}$$

For the lowest frequency (m = 1), we obtain $(\lambda/2a) = 1.08$. Thus our analysis for the bubble yields a lower frequency than does Lamb's analysis for a gas-filled rigid sphere ($\lambda/2a = 0.699$), as expected.

C. Incompressible Bubble Limit

For oscillations of low enough frequency, the mass of pressurizing gas would be expected to flow as an incompressible fluid against the restoring force of surface tension. The question to be answered here is whether the frequency of assumed incompressible oscillations is low enough to justify the assumption.

An incompressible fluid may be defined as one for which the speed of sound is infinitely greater than the particle velocity of the fluid itself. Hence returning to equation (35a), we need only evaluate the right hand side in the limit $k \to 0$. From the series expansion of the spherical Bessel functions we find

$$\frac{z\,\psi_n'(z)}{\psi_n(z)} \sim -\frac{z^2}{2n+3} \xrightarrow[z \to 0]{} 0 \quad .$$

Hence we obtain the characteristic frequency for an incompressible droplet in a vacuum:

$$\left(\bar{\rho}\,\frac{a^3\,\omega^2}{T}\right) = 2\,n\,(n-1)\,(n+2)$$

Noting again that our T is the surface tension for only one side of the liquid film (accounting of the factor of 2), this formula is seen to agree exactly with Lamb's formula for a liquid drop. We note that oscillations are not possible for n = 0 (pure radial motion) because of the infinite stiffness, nor for n = 1 (pure lateral translation) because of the lack of a restoring force. Now let us regard the formula as applying to low frequency oscillations of a gas. Again using (35b), we have

$$\left(\frac{\omega a}{c}\right) = \sqrt{\frac{n(n-1)(n+2)}{2\gamma}}$$

The lowest frequency possible is that for n = 2. The wave length for this frequency is

$$\frac{\lambda}{2a} = \frac{\pi\gamma}{4} \sim 1$$

From this result we conclude that the incompressible model of bubble oscillation is not valid for the conditions considered here: namely, mass of bubble film negligible compared with that of the gas. Since $\omega \sim 1/\sqrt{m}$, it appears that the incompressible model is applicable only for film mass of the order of 100 times the mass of enclosed gas. This condition, which would exist only during the early stages of the inflation process, will be analyzed in the following section.

V. LOW FREQUENCY INSTABILITY OF INFLATION PROCESS

In the last section, we considered neutral oscillations about the equilibrium state. Since no processes of dissipation or excitation were involved, these oscillations are neutrally stable. However, the inflation process is inherently different, since the introduction of gas may serve as a source of energy for unstable perturbations from the desired bubble growth history.* In this concluding section of our study, we consider the problem of low-frequency instability. Besides simplifying the analysis considerably, as already shown in the preceding section, it is felt that restriction to low frequencies places the emphasis on the dominant mode of instability, since initially the actual film material exhibits strongly viscous properties (although here approximated as inviscid), which should severely damp high frequency oscillations. As shown previously, low frequency oscillations can occur only during the early stage of the inflation process while the film is relatively massive compared with the gas pressurant.

* A source of energy for growth of an unstable mode is the distinguishing feature between passive and active systems and their well-known dynamic characteristics.

A. Related Literature

In recent years, some effort has been devoted to bubble dynamics because of interest in several unrelated fields, among which we mention underwater explosions and boiling heat transfer. Perhaps the first analysis of significance is that of Taylor (Ref. 5), who considered a plane interface between two fluids of different densities. His analysis showed that instability resulted if the interface accelerates toward the medium of higher density. This type of instability is now regarded as "classic" and is one of several called "Taylor instability". In 1953 Binnie (Ref. 6) applied the methods of Rayleigh and Lamb, for the vibration of a liquid drop, to the new problem of stability of an expanding bubble. His analysis showed that a growing bubble is unstable, a shrinking bubble stable, and that the instability is of the type found by Taylor. Binnie assumed the perturbations to be exponentially time dependent, which limits his results to the initial phases of the instability and slow expansions; more significant, however, is that the boundary conditions were evaluated at the interface of the unperturbed bubble, whereas the perturbed surface must be considered for a growing bubble. The problem was reformulated by Plesset (Ref. 7), who corrected the error of Binnie. In the same year Birkhoff (Ref. 8) carried out an asymptotic analysis of a collapsing bubble without surface tension and concluded that the collapsing bubble is unstable for sufficiently small radius. Neither of these analyses were limited to exponential time dependence. Plesset and Mitchell (Ref. 9) carried out analytical solutions for arbitrary radius, with and without surface tension, based on the earlier formulation of Plesset, Their analysis, neglecting the fluid density on the interior of the bubble, showed that a perturbation initially grows for the case of an expanding bubble but not for a collapsing bubble (except near zero radius). However, when the perturbation amplitude is normalized with respect to the mean bubble radius, the perturbation remains finite for all time. These conclusions were reached for the special case of a bubble growing at constant internal pressure, corresponding to a vapor bubble in a liquid bath. For other types of bubbles, it is conceivable that different conclusions would result.*

B. Dynamical Equations for Thick-Walled Bubble

The analyses described above were primarily concerned with thin vapor bubbles immersed in an infinite liquid bath. In contrast, the problem of concern to us includes both thick and thin-walled bubbles being blown at an arbitrary rate in a vacuum

* Cole (Ref.10) reports on numerical solutions by Penney and Price for a pulsating gas sphere immersed in a liquid. The internal pressure was given by the isentropic pressure-volume relationship, corresponding to a constant mass of internal gas. Their results were in qualitative agreement with Plesset and Mitchell, showing the largest perturbations near zero radius.

environment. In this section, the analysis is formulated for thick-walled bubbles subject to low frequency oscillations about a slowly varying radius. With these restrictions, the pressurizing gas may be treated as an incompressible fluid, simplifying the analysis to a great degree. In the following development, the bubble will be assumed to expand into an external atmosphere; this additional effect will be important in laboratory simulation.

Let the shape of the inner film surface be given by

$$r_1 = a_1(t) + \delta_1(t) S_n(\theta, \lambda) \tag{40a}$$

and that of the outer surface by

$$r_2 = a_2(t) + \delta_2(t) S_n(\theta, \lambda) \tag{40b}$$

where a_1 and a_2 are slowly changing functions of time and S_n is a surface harmonic as introduced in Section IV. Now the velocity potential, satisfying Laplace's equation, can be written as the sum of two terms:

$$\phi = \Phi + \phi^1 \tag{41}$$

where Φ represents the radial expansion and ϕ^1 the oscillatory component. Then conservation of mass requires

$$\Phi = -\frac{Q}{4\pi r} = -\frac{a_1^2 \dot{a}_1}{r} = \frac{a_2^2 \dot{a}_2}{r} \tag{42}$$

The perturbation potential in terms of spherical harmonics is given by the expressions (also satisfying Laplace's equation):

$$\phi^1 = A(t)\, r^n\, S_n(\theta,\lambda) \qquad \text{for } r \le r_1 \tag{43a}$$

$$\phi^1 = \left[B(t)\, r^n + C(t)\, r^{-n-1}\right] S_n(\theta,\lambda) \quad \text{for } r_1 \le r \le r_2 \tag{43b}$$

$$\phi^1 = D(t)\, r^{-n-1}\, S_n(\theta,\lambda) \qquad \text{for } r \ge r_2 \tag{43c}$$

The parameters A, B, C, D, must be chosen to satisfy the boundary conditions. At the perturbed inner surface the velocity must be continuous and equal to the derivative of (40a):

$$\left(\frac{\partial\phi}{\partial r}\right)_{r_1} = \frac{d\, r_1}{d\, t} = a_1 + \delta_1 S_n(\theta,\lambda)$$

Noting that

$$\left(\frac{\partial\Phi}{\partial r}\right)_{r_1} = \frac{a_1^2\, \dot{a}_1}{\left(a_1+\delta_1 S_n\right)^2} = \dot{a}_1 \left[1 - 2\left(\frac{\delta_1}{a_1}\right) S_n + \ldots\right]$$

we find (to first order in δ)

$$A = \frac{1}{n\, a_1^n} \left[2\, \dot{a}_1\, \delta_1 + a_1\, \dot{\delta}_1\right] \tag{44a}$$

while continuity of velocity at r_1 requires

$$B = \frac{n+1}{n}\, a_1^{-2n-1} \qquad C = A \tag{44b}$$

At the outer perturbed surface we require

$$\left(\frac{\partial\phi}{\partial r}\right)_{r_2} = \frac{d\, r_2}{d\, t} = a_2 + \delta_2\, S_n(\theta,\lambda)$$

Proceeding as above we find

$$n\, a_2^n\, B - (n+1)\, a_2^{-n-1}\, C = -(n+1)\, a_2^{-n-1}\, D \tag{44c}$$

and

$$D = -\frac{a_2^{n+1}}{n+1}\left[2\, \dot{a}_2\, \delta_2 + a_2\, \delta_2\right] \tag{44d}$$

The remaining two conditions relate the pressure jump due to surface tension at each of the two surfaces to the curvature at each surface. For unsteady flow, the Bernoulli equation is

$$p + q + \rho \frac{\partial \phi}{\partial t} = f(t) \ , \quad q = \frac{1}{2} \rho \left(\frac{\partial \phi}{\partial r}\right)^2$$

(Note that the time dependent part of f(t) may be absorbed in ϕ with no loss of generality.) Following Lamb (Ref. 4), the pressure jump across each interface is given by

$$\Delta p = T \left\{ \frac{2}{a} + \frac{(n-1)(n+2)}{a^2} \delta S_n \right\}$$

Evaluating each term of the unsteady Bernoulli equation from (42) and (43) and expanding non-linear terms in δ (retaining only first order terms), the pressure jump at r_1 is given by

$$\underbrace{(\rho_F - \rho_G) \frac{\dot{Q}\delta_1}{4\pi a_1^2} - \rho_G a_1^n \dot{A} + \rho_F a_1^n \dot{B} + \rho_F a_1^{-n-1} \dot{C}}_{\frac{\partial \phi}{\partial t}} + \underbrace{(\rho_F - \rho_G)}_{q} \dot{a}_1 \dot{\delta}_1 = \underbrace{(n-1)(n+2)}_{\text{surface tension}} \frac{T\delta_1}{a_1^2}$$

Similarly, the pressure jump condition at the outer surface becomes

$$(\rho_F - \rho_e) \frac{\dot{Q}\delta_2}{4\pi a_2^2} + \rho_F a_2^n \dot{B} + \rho_F a_2^{-n-1} \dot{C} - \rho_e a_2^{-n-1} \dot{D} + (\rho_F - \rho_e) \dot{a}_2 \dot{\delta}_2 = -(n-1)(n+2) \frac{T\delta_2}{a_2^2} \qquad (44f)$$

Equations (44) provide six linear differential equations for the six functions A, B, C, D, δ_1, and δ_2. In general, these equations cannot be uncoupled in a simple manner. Hence we shall not attempt to solve them exactly. Instead we shall first seek an approximate solution, based on a quasi-steady state assumption, for slowly expanding bubbles. Following this, an exact solution of the equations will be sought for cases in which the bubble film is sufficiently thin that flow within the film may be neglected, although the mass of the film will be retained. The latter analysis is thus an extension of Plesset's analysis for a simple interface.

C. Quasi-steady Analysis for Thick-walled Bubbles

Let us now assume that the mean bubble radius changes very slowly. Then we may regard the coefficients in equations (44a) - (44f) as constants over a few cycles of oscillation of the perturbation. With this assumption, we look for solutions as complex exponential functions of time:

$$\delta_1 = \bar{\delta}_1 e^{i\omega t}, \ \delta_2 = \bar{\delta}_2 e^{i\omega t}, \ A = \bar{A} e^{i\omega t}, \ B = \bar{B} e^{i\omega t}, \ C = \bar{C} e^{i\omega t}$$

where the barred quantities are slowly varying functions of time, treated as constants. Then whenever a time derivative of one of these variables occurs, it may be replaced by $(i\omega)$ times that variable. In this way we obtain a set of linear algebraic equations in the barred quantities. The system can be reduced to three equations if $\dot{A}$, $\dot{B}$, and D are eliminated by use of (44a), (44b), and (44c):

$$\left\{(n+1)\, a_2^{-n-1}\left[\left(\frac{a_2}{a_1}\right)^{2n+1}-1\right]\right\}\bar{C}+\left\{\left(\frac{a_2}{a_1}\right)^{n}\left(2\dot{a}_1+i\omega a_1\right)\right\}\delta_1-\left(2\dot{a}_2+i\omega a_2\right)\delta_2=0 \qquad (45a)$$

$$\left\{\left(\frac{2n+1}{n}\right)i\omega\rho_F\, a_1^{-n-1}\right\}\bar{C}+\left\{-(n-1)(n+2)\left(\frac{T}{a_1^2}\right)+\left(\rho_F-\rho_G\right)\left[\frac{\dot{Q}}{4\pi a_1^2}+\frac{n+2}{n}\, i\,\omega\,\dot{a}_1-\frac{\omega^2 a_1}{n}\right]\right\}\delta_1=0 \qquad (45b)$$

$$\left\{\left[\left(\frac{n+1}{n}\right)\left(\frac{a_2}{a_1}\right)^{2n+1}+1\right]i\omega\rho_F\, a_2^{-n-1}\right\}\bar{C}+\left\{\frac{i\omega\rho_F}{n}\left(\frac{a_2}{a_1}\right)^{n}\left(2\dot{a}_1+i\omega a_1\right)\right\}\delta_1$$
$$+\left\{(n-1)(n+2)\left(\frac{T}{a_2^2}\right)+\frac{i\omega\rho_e}{n+1}\left(2\dot{a}_2+i\omega a_2\right)+\left(\rho_F-\rho_e\right)\left[\frac{\dot{Q}}{4\pi a_2^2}+i\omega\dot{a}_2\right]\right\}\bar{\delta}_2=0 \qquad (45c)$$

Since the equations are homogeneous, the frequency ω is determined by the vanishing of the determinant of the system, in order to avoid a non-trivial solution.

Note that equation (45) contains eleven parameters: n, T, ω, a_1, a_2, $\dot{a}_1$, $\dot{a}_2$, ρ_F, ρ_G, ρ_e, $\dot{Q}$, so that a general solution by numerical methods appears formidable. However, the theory of dimensional analysis (Ref. 2) permits a considerable reduction in the number of independent variables. The parameter $\dot{a}_2$ may be eliminated by conservation of mass in the film ($a_2^2\dot{a}_2=a_1^2\dot{a}_1$). Thus with ten dimensional parameters in three independent dimensions (mass, length, time) we can have only 7 non-dimensional groups. We choose

$$n,\ \Omega=\frac{i\omega a_1}{\dot{a}_1},\ \tau=T/\rho_F\, a_1\, (\dot{a}_1)^2,\ \alpha={}^{a_2}/_{a_1},\ F=\dot{Q}/4\pi a_1(\dot{a}_1)^2,\ k_1={}^{\rho_G}/_{\rho_F},\ k_2={}^{\rho_e}/_{\rho_F} \qquad (46)$$

In these variables, the characteristic equation appears as

$$\begin{vmatrix} (n+1)\left(\alpha^{2n+1}-1\right) & n\alpha^{n}(\Omega+2) & -\left(\alpha^{3}\Omega+2\right) \\ \left(\frac{2n+1}{n}\right)\Omega\alpha^{n+1} & \left\{-n(n-1)(n+2)\tau+(1-k_1)\left[nF+(n+2)\Omega+\Omega^{2}\right]\right\} & 0 \\ \left[\left(\frac{n+1}{n}\right)\alpha^{2n+1}+1\right]\Omega & \Omega\alpha^{2}(\Omega+2) & \left\{(n-1)(n+2)\tau+(1-k_2)(F+\Omega) + \frac{1}{n+1}k_2\,\Omega\left(1+\Omega\alpha^{3}\right)\right\} \end{vmatrix} = 0 \tag{47}$$

Of special interest is the case n = 2 (lowest mode) which would be expected to be least affected by viscosity, and $k_1 = k_2 = 0$. Further if we consider a particular type of bubble growth, such as constant volume flux (Q = constant), then Q is given, and the problem is reduced to three variables:

$$\Omega = \Omega(\alpha, \tau)$$

Numerical calculations then become practical, and the results could be presented on a single graph; for example, a curve showing neutral stability on a graph of α versus τ.*
Note that the present theory would not be expected to hold for α near 1, as we have shown in a previous section that the film mass must be large to have low frequency oscillations.

In this section we have reduced the problem of stability of thick-walled bubbles to a straightforward numerical calculation of the complex frequency. Direct analytical results for

* By setting $\rho_F = 0$, the motion of the inner surface becomes uncoupled from that of the outer surface. The value of the determinant then becomes the product of the terms on the principal diagonal. The two factors containing surface tension then separately determine the stability of the two surfaces. For $\dot{a}_1 = \dot{a}_2 = 0$, the result reduces to that obtained by Lamb for a spherical drop (Ref. 4).

this case do not appear to be forthcoming. However, if the bubble film is thin, so that flow within the film is negligible, then an analytical approach including the effects of the mass of the film is possible. An analysis of this type is carried out as the final section of this report.

D. General Analysis of Thin-walled Bubbles

In making the thin-film approximation $r_2 \to r_1$, the variables of the fluid mechanical problem are greatly reduced in number. Thus dropping δ_2 , B and C from consideration leaves only three variables $\delta_1 (= \delta)$, A, and D, to be determined from (44a), (44d), and (44e). Actually (44e) gives only the pressure jump across the inner film surface; both surfaces are included by subtracting (44f) from (44e). In addition we include a term representing the pressure necessary to accelerate the mass of the film (let m_F be mass of film per unit surface area). With $a_2 \equiv a_1 \equiv a,\ \delta_2 \equiv \delta_1 \equiv \delta,\ B \equiv C \equiv O$, the pressure jump condition becomes

$$(\rho_e - \rho_G) \frac{\dot{Q}\delta}{4\pi a^2} + \rho_e a^{-n-1} D - \rho_G a^n A + (\rho_e - \rho_G) \dot{a}\delta = 2(n-1)(n+2) \frac{T\delta}{a^2} + m_F \delta \qquad (48a)$$

Now (44a) and (44d) take on the values

$$n a^n A = -(n+1) a^{-n-1} D = 2 \dot{a} \delta + a \dot{\delta} \qquad (48b)$$

Hence the time derivatives become

$$a^n A = \frac{1}{n} \left\{ a \delta - (n-3) a \delta - 2 \left[\frac{n a^2}{a} - a \right] \delta \right\} \qquad (48c)$$

$$a^{-n-1}\ D = -\frac{1}{n+1}\left\{a\ \ddot{\delta} + (n+4)\ a\,\delta + 2\left[(n+1)\ \frac{a^2}{a} + a\right]\delta\right\} \tag{48d}$$

Also by (42)

$$\frac{Q}{4\pi} = \frac{d}{dt}\left(a^2\ a\right) = a^2\ a + 2\ a\ a^2 \tag{48e}$$

With these additional relations, (48a) becomes a second order differential equation for δ :

$$\ddot{\delta} + \frac{3}{\beta}\left(\frac{a}{a}\right)\delta + \frac{\alpha}{\beta}\ \delta = 0 \tag{49a}$$

where

$$\alpha = \frac{2n(n-1)(n+1)(n+2)(T/a^2)-[n(n-1)\rho_e - (n+1)(n+2)\rho_G]\ a}{[n\rho_e + (n+1)\rho_G]\ a} \tag{49b}$$

$$\beta = 1 + \frac{n(n+1)\ m_F}{[n\rho_e+(n+1)\rho_G]\ a} \tag{49c}$$

Note that both α and β vary with time.

Aside from the film mass parameter β which we have introduced, this equation is identical to the one derived by Plesset for the stability of vapor bubbles. Note that our surface tension is twice that of Plesset; the difference is caused by the occurrence of a foreign material in the film of our problem, whereas Plesset considered only an interface between the internal and external fluids.

The differential equation (49) takes on a simpler form if a new dependent variable is used. Let

$$\Delta(t) = \delta(t)\,(a/a_0)^{3/2\beta} \tag{50}$$

where a_0 is some constant dimension, characteristic of the bubble. Then (49a) is transformed to

$$\ddot{\Delta} + \left(\frac{\alpha - g}{\beta}\right)\Delta = 0 \tag{51a}$$

where

$$g(t) = \frac{3}{2}\left[\left(\frac{3}{2\beta} - 1\right)\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right] \tag{51b}$$

Equation (51) has a form which lends itself to a "static stability" analysis: the criterion for stability is that the acceleration be opposite to the displacement.*

$$\alpha > g \quad : \quad \text{stable condition} \tag{52a}$$

$$\alpha < g \quad : \quad \text{unstable condition} \tag{52b}$$

For equations with constant coefficients, we know that (52a) corresponds to purely sinusoidal motion as opposed to exponential behavior for (52b). Let us now consider three examples.

<u>Example 1</u>: Bubble Expending at Uniform Rate in Vacuum

For this case we have $\rho_e = 0$

$\dot{a} = 0$, $a = \text{constant}$, $a = a\,t$.

* The more general criteria are given by Birkhoff (Ref. 11).

With these values we find

$$\alpha = 2n(n-1)(n+2)\left(\frac{T}{\rho_G a^3}\right)$$

$$\beta = 1 + \left(\frac{n\,m_F}{\rho_G a}\right)$$

$$g = \frac{3}{2}\left(\frac{3}{2\beta} - 1\right)\left(\frac{\dot{a}}{a}\right)^2$$

Now for small a,

$$\beta \sim \frac{n\,m_F}{\rho_G a}, \quad g \sim -\frac{3\,a^2}{2\,a^2}$$

Hence for sufficiently small a, $\alpha - g > 0$ and the motion is stable. Similarly, for large a

$$\beta \sim 1, \quad g \sim \frac{3\,\dot{a}^2}{4\,a^2}$$

Hence for sufficiently large a, $\alpha - g < 0$ and the motion is unstable. The latter case, for which the mass of the film and surface tension both become negligible, was treated by Plesset (Ref. 6). This instability is not of the Taylor type since $\dot{a} = 0$.

<u>Example 2:</u> Bubble Expanding with Constant Mass Flux in Vacuum

In the preceding example the mass flux of pressurizing gas must increase rapidly with time, so that the energy supplied to the bubble also increases. Let us now consider a more likely case for which the mass flux of gas is held constant.

The total mass of gas is $M_o = 4\pi\rho_G a^3/3$ and so the mass flux is

$$\dot{M} = 4\pi \rho_G a^2 \dot{a}$$

Hence

$$a = \left(\frac{3\, Mt}{4\, \pi \rho_G}\right)^{1/3}$$

and

$$(a/a) = 1/3t \;, \quad a/a = -\,2/9t^2 \quad .$$

With these values, and with $\rho_e = 0$, we find

$$\alpha = n(n-1)(n+2) \left(\frac{8\pi T}{3\dot{M}t}\right) - \frac{2(n+2)}{9t^2}$$

$$\beta = 1 + n\, m_F \left(\frac{4\pi}{3\rho_G^2 Mt}\right)^{1/3}$$

$$g = \frac{1}{6t^2} \left(\frac{3}{2\beta} - 2\right)$$

Now for small time (small radius)

$$\alpha \sim \frac{2(n+2)}{9\, t^2} \;, \qquad g \sim -\frac{1}{3\, t^2}$$

so that for sufficiently small bubbles

$$\alpha - g \sim -\frac{2n+1}{9\, t^2}$$

and the motion is unstable. Similarly for large time (large radius)

$$\alpha \sim n(n-1)(n+2) \left(\frac{8\pi T}{3\dot{M}t}\right), \qquad g \sim -\frac{1}{12\, t^2}$$

Hence for sufficiently large bubbles $\alpha - g > 0$ and the motion is stable.

We see that the results for this example are exactly opposite those of the first example. It appears that the instability for large bubbles is associated with an increasing rate of energy supply, as was the case in the first example. Note that these results have been obtained for an inviscid fluid, and that viscous effects may alter the results, especially at the early stages of inflation.

Example 3: Bubble Oscillating about Equilibrium in Vacuum

Let us now consider the case in which a bubble has expanded beyond its equilibrium radius, and consequently experiences an oscillatory motion of the mean radius. For simplicity, we assume

$$a = a_o \left[1 + \epsilon \cos \omega t\right]$$

$$a = - a_o \epsilon \omega \sin \omega t$$

$$a = - a_o \epsilon \omega^2 \cos \omega t$$

Now we regard the amplitude of the spherical oscillation as small compared with the radius, and, in turn, the perturbation amplitude an order of magnitude smaller:

$$|\delta| << \epsilon << a$$

Then we can linearize the equation with respect to ϵ . Putting $\beta = 1$ (since bubble is large) we have

$$\alpha - g = \omega^2 (a' + q' \cos \omega t)$$

where

$$a' = 2n(n-1)(n+2) \left(\frac{T}{\rho_G a_o^3 \omega^2} \right) \tag{53a}$$

$$q' = - \left[3a' + (n + \frac{1}{2}) \right] \epsilon \tag{53b}$$

Defining the new variable $z = \omega t$, we find that our perturbation equation takes on the standard form of the Mathieu equation:

$$\frac{d^2 \Delta}{d z^2} + (a' + q' \cos z) \Delta = 0 \tag{53c}$$

The stability characteristics of Mathieu's equation are discussed by Stoker (Ref. 12) as well as many other authors. For small q', the solution is stable almost everywhere, with small regions of instability located in the vicinity of the points $a' = k^2/4$, $q' = 0$, where k is an integer. For larger q', these regions of instability increase in size until the solution is unstable almost everywhere for $q' >> a'$. Complete stability diagrams are readily available (see Ref. 13, for example) so that the stability of a bubble is easily determined for particular cases.* For unstable cases, our equation (53) ceases to be valid when the perturbation amplitude becomes comparable with the original amplitude of oscillation ϵ. The general equation (51b) must be applied then; note however, that an analysis such as that of Examples 1 and 2 fails for this case since $(a' + q' \cos z)$ can be always positive and yet have instability for Mathieu's equation. Thus the "static stability" analysis should be used with reservations. (The defects of this

* Note that several notations are used for Mathieu functions Thus Refs 12 and 13 use different definitions of the same symbol.

method have been pointed out earlier by Birkoff (Ref. 11). For example, bubbles with monotonically varying radius should be appropriate for a static stability investigation.

VI. CONCLUDING REMARKS

In this study we have attempted to investigate some of the fluid-dynamics problems concerned with blowing bubble-type enclosures in outer space. As a preliminary study, these problems were treated using a variety of assumptions and approximations in order to determine in a qualitative way the conditions under which different physical effects are important. In certain cases, the analysis was formulated but no solution was obtained, as in the case of self-similar analysis of the motion of the pressurizing gas. Here the need for further work is apparent so that the accuracy of more approximate methods can be assessed. In other cases, the analysis was carried out in sufficient detail to enable definite conclusions to be drawn, such as that of Section IV in which it was shown that an incompressible analysis is valid only for thick-walled bubbles. Thus, the author hopes that this report will serve to clarify the role of the various physical effects and to stimulate further work in this area.

The major assumption made throughout this study is that of a non-viscous fluid. Since the presence of viscosity should act to damp out any perturbing motion of the film*, our inviscid analysis

* We assume that viscosity does not play a subtle role as in fluid boundary layers, where the flow is first destabilized and then stabilized again as the viscosity increases.

is certainly restricted to relatively low frequencies. The determination of the quantitative effect of viscosity on the instabilities is perhaps the most important of the problems which we leave for future work.

Finally, it is in order to consider the practical realization for the proposed inflation process as a means of forming large-surfaced space devices. Such devices may be spherical shells, capable of reflecting electromagnetic radiation, such as the passive communication satellite. Other applications may involve the determination of high altitude properties of planetary atmospheres. For this purpose, a lightweight, high drag object can be ejected from an orbiting exploratory vehicle. Optical or radar tracking of the object will then yield data on atmospheric density, winds, etc. To obtain meaningful data, it will be required that the ejected object be large, of spherical symmetry, and of high drag/weight ratio to allow tracking and to prevent premature destruction by aerodynamic heating (Ref. 14).

Two possible concepts of forming extremely large diameter and extremely lightweight spheres in space are shown in Figure 4. One consists of an extrusion - inflation device similar to those used in industrial plastic film forming equipment. It comprises a cylinder piston assembly, filled with the liquid material, activated by pneumatic pressure. Some additional details of this concept are shown in Figure 5. The other concept involves a free-sphere of the liquid, with the inflation device placed in the center. For both means of inflation, solvent evaporation from the inside surface of the liquid can be used as a portion of the inflation gas source.

Vacuum deposition of a thin metal film on the sphere's inside after completion of the inflation process can be considered for applications requiring radar reflectivity. Exploratory experimentation conducted during this study shows the basic feasibility of this approach.

A critical problem that remains to be studied experimentally is the effect of vacuum on viscosities, hardening rates, evaporation rates and surface tensions of candidate liquids. Viscosity and surface tension need to be kept sufficiently low to allow inflation with a minimum of internal gas mass; hardening rates need to remain sufficiently low to permit adequate time for the inflation process. Quantitative data can be obtained by fairly simple laboratory investigations. These studies also will provide data for the analytical process stability investigation including viscosity effects proposed above.

The results of such a combined experimental - analytical study should provide criteria for the inflation gas flow rate control system which will be required to insure process stability.

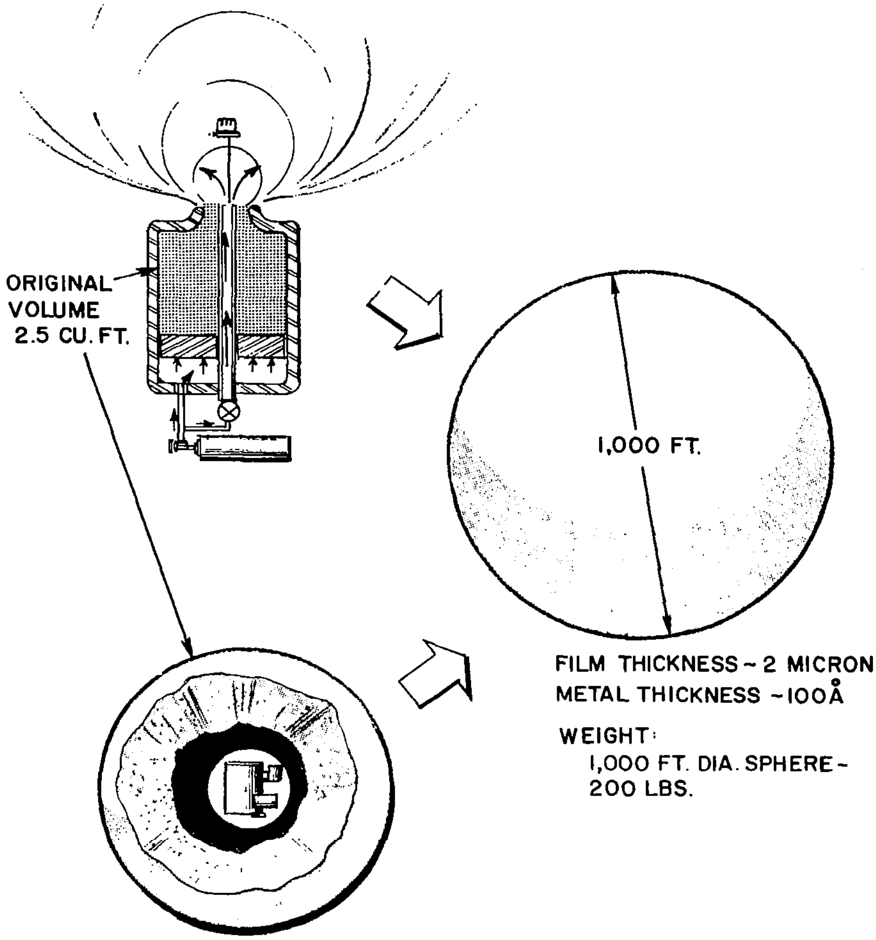

Figure 4. Forming Concept

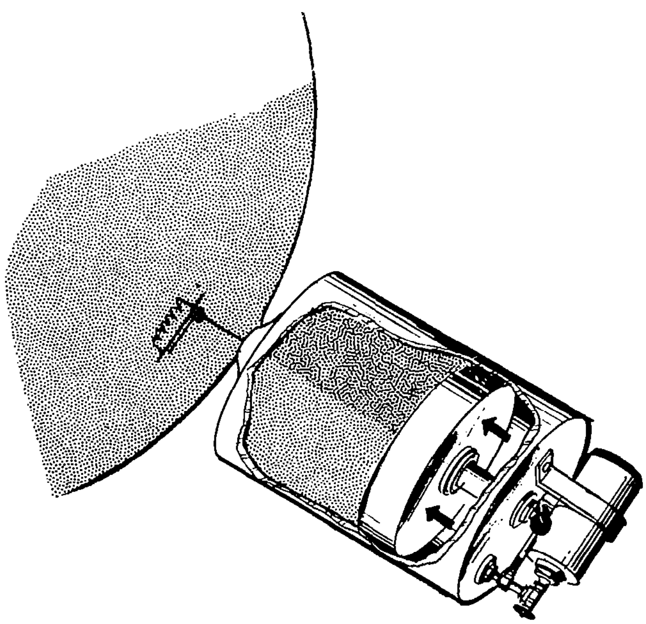

Figure 5. Extrusion - Inflation Device

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